Chapter 4
Introduction to Probability

- Experiments, Counting Rules, and Assigning Probabilities
- Events and Their Probabilities
- Some Basic Relationships of Probability
- Conditional Probability
- Bayes' Theorem

Uncertainties

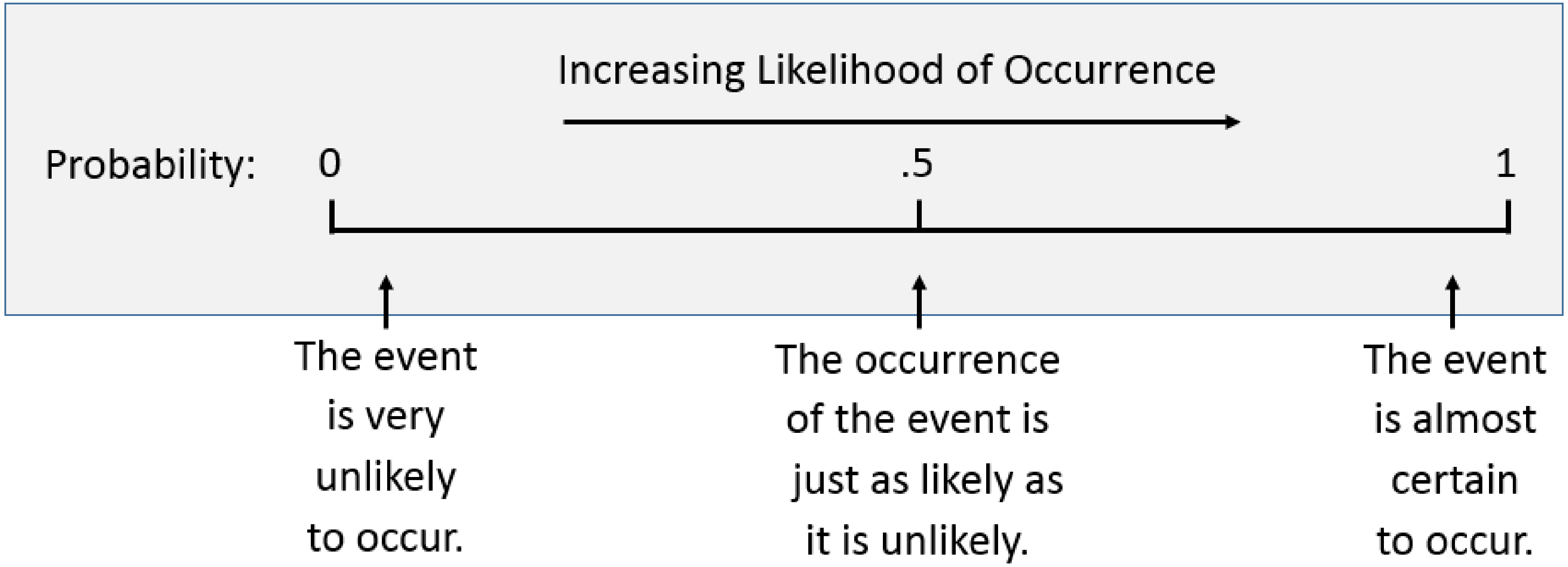
Managers often base their decisions on an analysis of uncertainties such as the following:

- What are the *chances* that sales will decrease if we increase prices?
- What is the *likelihood* a new assembly method will increase productivity?
- What are the *odds* that a new investment will be profitable?

Probability

- Probability is a numerical measure of the likelihood that an event will occur.
- Probability values are always assigned on a scale from 0 to 1.
  - A probability near zero indicates an event is quite unlikely to occur.
  - A probability near one indicates an event is almost certain to occur.

Probability as a Numerical Measure of the Likelihood of Occurrence



Statistical Experiments

- In statistics, the notion of an experiment differs somewhat from that of an experiment in the physical sciences.
- In statistical experiments, probability determines outcomes.
- Even though the experiment is repeated in exactly the same way, an entirely different outcome may occur.
- For this reason, statistical experiments are sometimes called *random experiments.*

An Experiment and Its Sample Space (1 of 2)

- An experiment is any process that generates well-defined outcomes.
- The sample space for an experiment is the set of all experimental outcomes.
- An experimental outcome is also called a sample point.

An Experiment and Its Sample Space (2 of 3)

Experiment	Experiment Outcomes
Toss a coin	Head, tail
Inspection a part	Defective, non-defective
Conduct a sales call	Purchase, no purchase
Roll a die	1, 2, 3, 4, 5, 6
Play a football game	Win, lose, tie

An Experiment and Its Sample Space (3 of 3)

Example: Kentucky Power and Light Company (KP&L)

KP&L has designed a project to increase the generating capacity of one of its plants. The project is divided into two sequential stages: Stage 1 – Design; Stage 2 – Construction. The possible completion times of these stages are as follows:

Possible completion time in months	
Design Stage	Construction Stage
2	6
3	7
4	8

A Counting Rule for Multiple-Step Experiments (1 of 2)

- If an experiment consists of a sequence of *k* steps in which there are

  n_1 possible results for the first step

  n_2 possible results for the second step, and so on.

  Then the total number of experimental outcomes is given by

$$(n_1)(n_2)\ldots(n_k).$$

- A helpful graphical representation of a multiple-step experiment is a tree diagram.

A Counting Rule for Multiple-Step Experiments (2 of 2)

Example: Kentucky Power and Light Company (KP&L)

(KP&L) expansion project can be viewed as a two-step experiment. It involves two stages, each of which have three possible completion times.

Design stage $n_1 = 3$

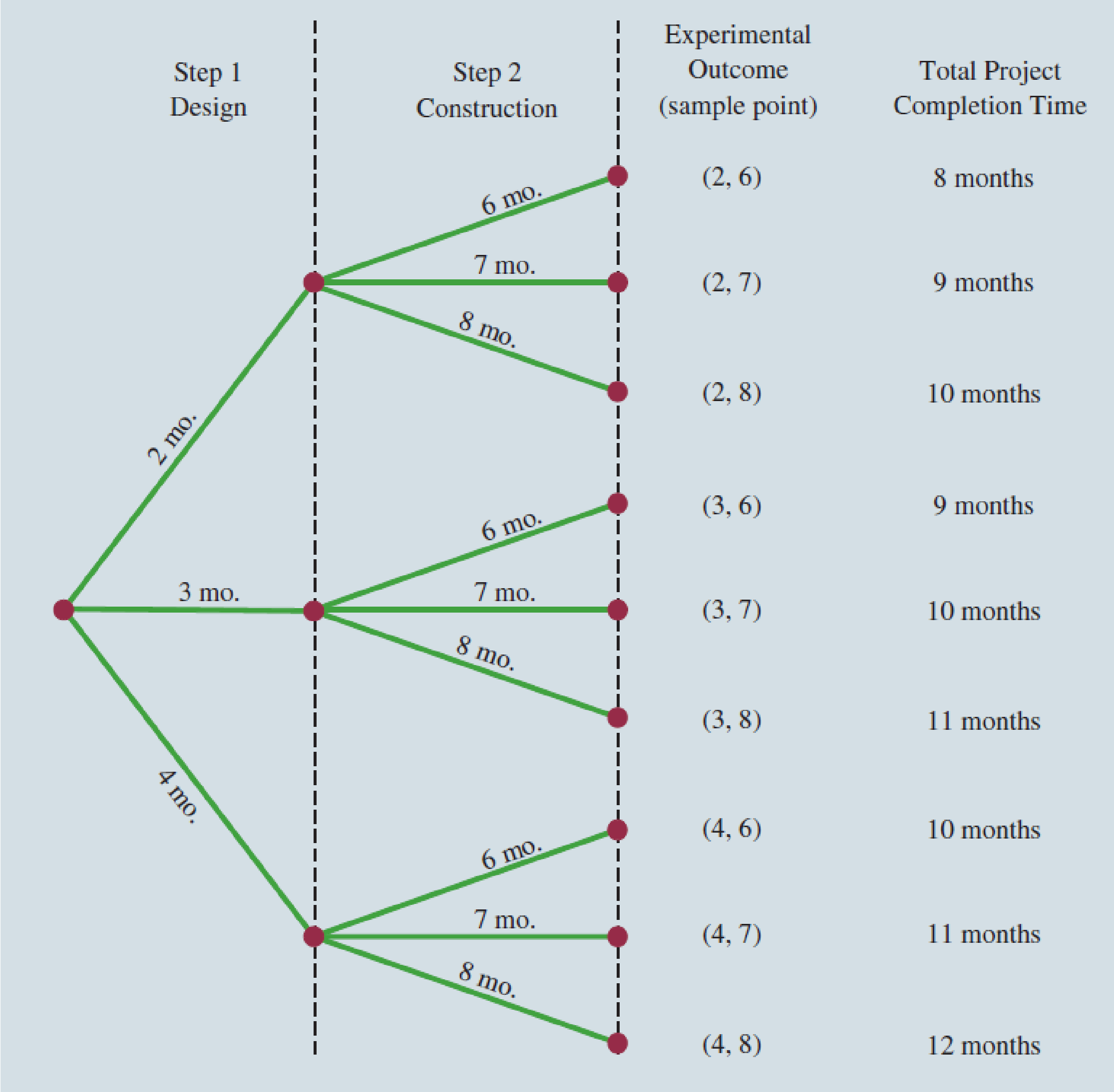
Construction stage $n_2 = 3$

Total Number of Experimental Outcomes: $n_1 n_2 = (3)(3) = 9$

Tree Diagram (1 of 2)

Example:

Kentucky Power and Light Company (KP&L)



Counting Rule for Combinations

Number of Combinations of N objects taken n at a time

A second useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects.

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

$$\text{where}: N! = N(N-1)(N-2)\ldots(2)(1)$$

$$n! = n(n-1)(n-2)\ldots(2)(1)$$

$$0! = 1$$

Counting Rule for Permutations

Number of Permutations of N Objects Taken n at a Time

A third useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects, where the order of selection is important.

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$$

$$\text{where}: N! = N(N-1)(N-2)\ldots(2)(1)$$

$$n! = n(n-1)(n-2)\ldots(2)(1)$$

$$0! = 1$$

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Assigning Probabilities (1 of 3)

- Basic Requirements for Assigning Probabilities

  1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively.

$$0 \leq P(E_i) \leq 1 \text{ for all } i$$

where E_i is the ith experimental outcome

$P(E_i)$ is its probability

Assigning Probabilities (2 of 3)

- Basic Requirements for Assigning Probabilities

  2. The sum of the probabilities for all experimental outcomes must equal 1.

$$P(E_1) + P(E_2) + \ldots + P(E_n) = 1$$

where: n is the number of experimental outcomes

Assigning Probabilities (3 of 3)

- Classical Method

  Assigning probabilities based on the assumption of equally likely outcomes.

- Relative Frequency Method

  Assigning probabilities based on experimentation or historical data.

- Subjective Method

  Assigning probabilities based on judgment.

Classical Method

Example: Rolling a Die

If an experiment has *n* possible outcomes, the classical method would assign a probability of $1/n$ to each outcome.

Experiment: Rolling a die

Sample Space: S = {1, 2, 3, 4, 5, 6}

Probabilities: Each sample point has a 1 in 6 chance of occurring

Relative Frequency Method (1 of 2)

Example: Waiting time in the X-ray department for a local hospital

Consider a study of waiting times in the X-ray department for a local hospital. A clerk recorded the number of patients waiting for service at 9:00 A.M on 20 successive days and obtained the following results:

Number waiting	Number of days outcome occurred
0	2
1	5
2	6
3	4
4	3
Total	**20**

Relative Frequency Method (2 of 2)

Example: Waiting time in the X-ray department for a local hospital

Each probability assignment is given by dividing the frequency (number of days) by the total frequency (total number of days).

Number waiting	Number of days outcome occurred	Probability
0	2	2/20 = 0.1
1	5	0.25
2	6	0.30
3	4	0.20
4	3	0.15
Total	**20**	**1.00**

Subjective Method (1 of 3)

- When economic conditions or a company's circumstances change rapidly it might be inappropriate to assign probabilities based solely on historical data.
- We can use any data available as well as our experience and intuition, but ultimately a probability value should express our degree of belief that the experimental outcome will occur.
- The best probability estimates often are obtained by combining the estimates from the classical or relative frequency approach with the subjective estimate.

Subjective Method (2 of 3)

Example:

Tom and Judy make an offer to purchase a house.

Possible outcomes: $E_1 =$ offer is accepted

$E_2 =$ offer is rejected

Judy's Probability estimates	Tom's probability estimates
$P(E_1) = 0.8$	$P(E_1) = 0.6$
$P(E_2) = 0.2$	$P(E_2) = 0.4$

Subjective Method (3 of 3)

Example:

- The assigned probabilities satisfy the two basic requirements.
- The different probability estimates emphasize the personal nature of the subjective method.

Events and Their Probabilities (1 of 4)

- An event is a collection of sample points.
- The probability of any event is equal to the sum of the probabilities of the sample points in the event.
- If we can identify all the sample points of an experiment and assign a probability to each, we can compute the probability of an event.

Events and Their Probabilities (2 of 4)

Example:

Kentucky Power and Light Company (KP&L)

We have completion results for 40 KP&L projects.

Completion time in months				
Design	Construction	Sample Point	Number of past projects with this completion time	Probability
2	6	(2,6)	6	0.15
2	7	(2,7)	6	0.15
2	8	(2,8)	2	0.05
3	6	(2,6)	4	0.10
3	7	(2,7)	8	0.20
3	8	(2,8)	2	0.05
4	6	(2,6)	2	0.05
4	7	(2,7)	4	0.10
4	8	(2,8)	6	0.15

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Events and Their Probabilities (3 of 4)

<u>Example:</u> Kentucky Power and Light Company (KP&L)

Event $C =$ Project will take 10 months or less to complete

$$C = \{(2,6),\ (2,7),\ (2,8),\ (3,6),\ (3,7),\ (4,6)\}$$

$$P(C) = \{P(2,6) + P(2,7) + P(2,8) + P(3,6) + P(3,7) + P(4,6)\}$$

$$P(C) = 0.15 + 0.15 + 0.05 + 0.10 + 0.20 + 0.05 = 0.7$$

Events and Their Probabilities (4 of 4)

Example: Kentucky Power and Light Company (KP&L)

Event $L =$ Project will take less than 10 months to complete

$$C = \{(2,6),\ (2,7),\ (3,6)\}$$

$$P(C) = \{P(2,6) + P(2,7) + P(3,6)\}$$

$$P(C) = 0.15 + 0.15 + 0.10 = 0.4$$

Some Basic Relationships of Probability

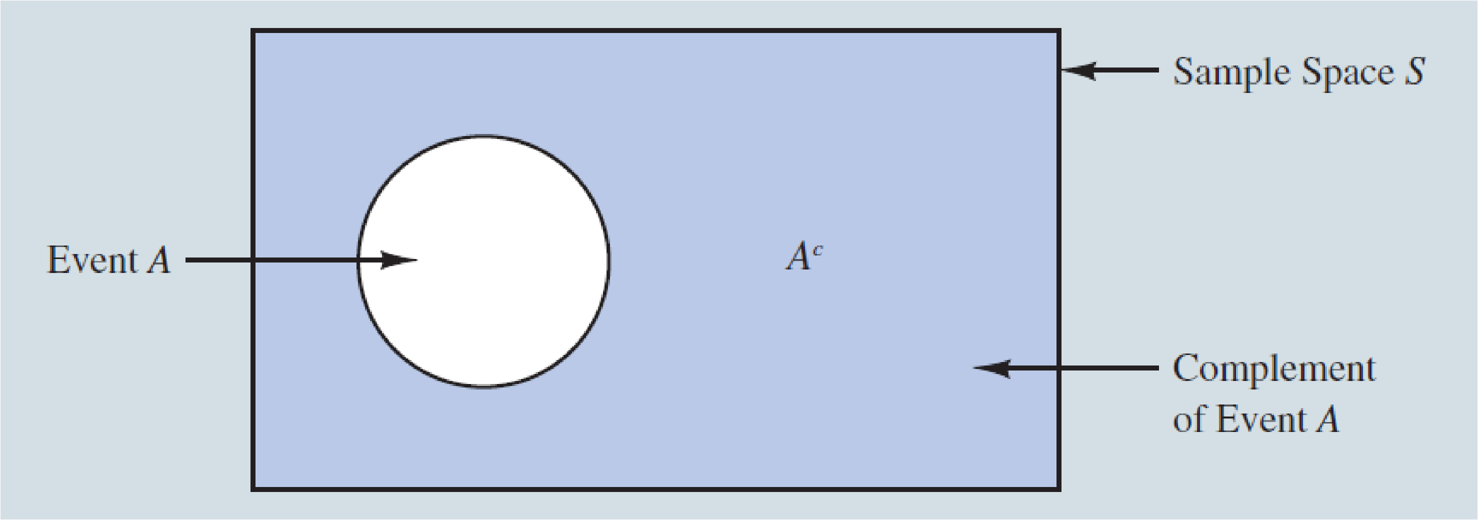
There are some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities.

- Complement of an Event
- Union of Two Events
- Intersection of Two Events
- Mutually Exclusive Events

Complement of an Event

- The complement of event A is defined to be the event consisting of all sample points that are *not* in A.
- The complement of A is denoted by A^c.

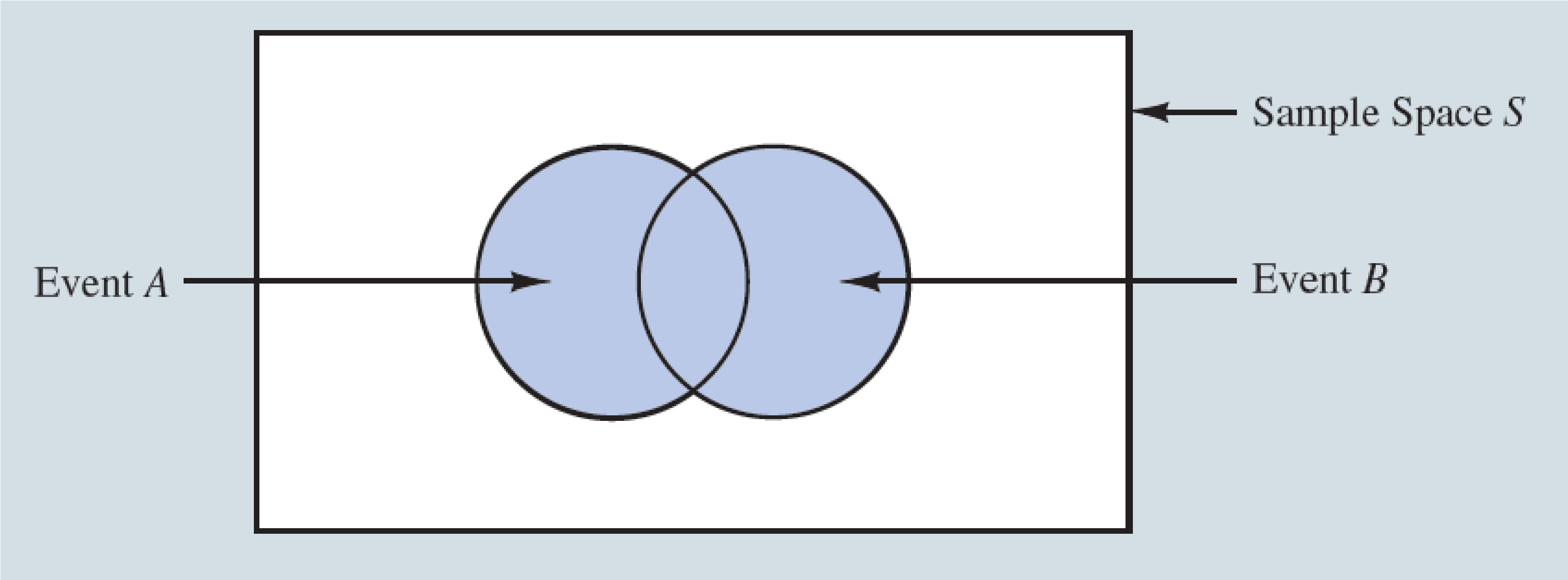
$$P(A) + P(A^c) = 1$$



Complement of event A is shaded.

Union of Two Events

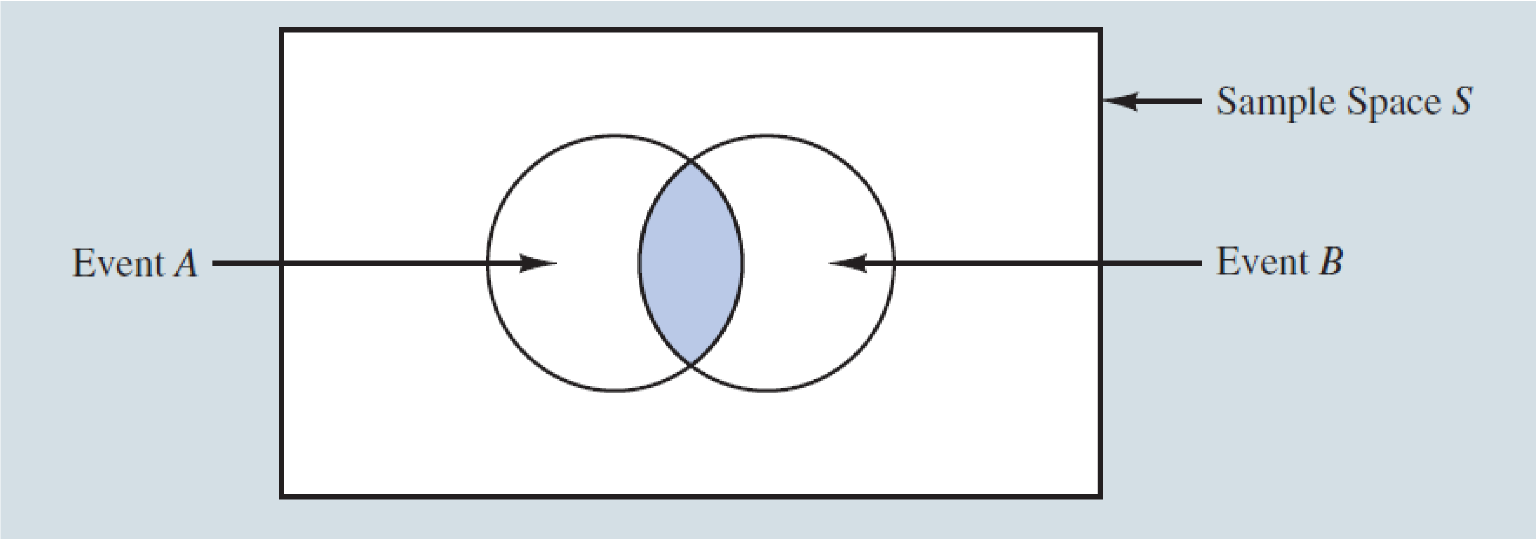
- The union of events *A* and *B* is the event containing all sample points that are in *A* or *B* or both.
- The union of events *A* and *B* is denoted by $A \cup B$.



Intersection of Two Events

- The <u>intersection</u> of events *A* and *B* is the set of all sample points that are in both *A* and *B*.
- The intersection of events *A* and *B* is denoted by $A \cap B$.

Intersection of *A* and *B*



Addition Law (1 of 4)

- The addition law provides a way to compute the probability of event *A*, or *B*, or both *A* and *B* occurring.
- The law is written as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Addition Law (2 of 4)

Example

A small assembly plant with 50 employees is carrying out performance evaluations. Each worker is expected to complete work assignments on time and in such a way that the assembled product will pass a final inspection. The results were as follows:

Result	Number of Employees	Relative Frequency
Late completion of work	5	5/50 = 0.1
Assembled a defective work	6	0.12
Completed work late and assembled defective work	2	0.04

Addition Law (3 of 4)

Example

Event L = the event that the work is completed late

Event D = the event that the assembled product is defective

The production manager decided to assign poor performance ratings to any employee whose work is either late or defective.

Addition Law (4 of 4)

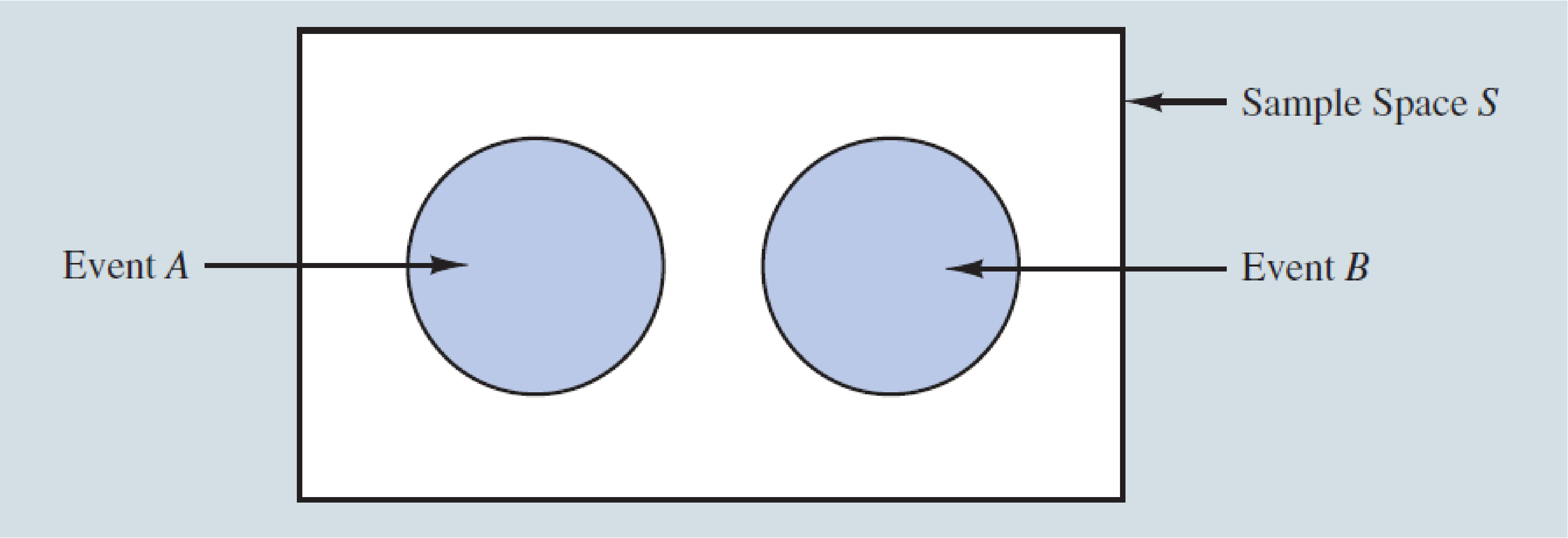
$L \cup D =$ Event that the production manager assigned an employee a poor performance rating

$$P(L \cup D) = P(L) + P(D) - P(L \cap D)$$

$$P(L \cup D) = 0.10 + 0.12 - 0.04 = 0.18$$

Mutually Exclusive Events (1 of 2)

- Two events are said to be mutually exclusive if the events have no sample points in common.
- Two events are mutually exclusive if, when one event occurs, the other cannot occur.



Mutually Exclusive Events (2 of 2)

- If events *A* and *B* are mutually exclusive, $P(A \cap B) = 0$.
- The addition law for mutually exclusive events is:

$$P(A \cup B) = P(A) + P(B)$$

Note: There is no need to include "$-P(A \cap B)$"

Conditional Probability (1 of 4)

- The probability of an event given that another event has occurred is called a conditional probability.
- The conditional probability of *A* given *B* is denoted by

$$P\left(A \middle| B\right)$$

- A conditional probability is computed as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability (2 of 4)

Example: Promotion status of police officers over the past two years

	Men	Women	Total
Promoted	288	36	**324**
Not Promoted	672	204	**876**
Total	**960**	**240**	**1200**

Conditional Probability (3 of 4)

Example: Promotion status of police officers over the past two years

Probability Table

	Men (M)	Women (W)	Total
Promoted (A)	$288/1200 = 0.24$	0.03	**0.27**
Not Promoted (A^c)	0.56	0.17	**0.73**
Total	**0.80**	**0.20**	**1.00**

Joint Probability

Joint probability table

	Men (M)	Women (W)	Total
Promoted (A)	$288/1200 = 0.24$	0.03	**0.27**
Not Promoted (A^c)	0.56	0.17	**0.73**
Total	**0.80**	**0.20**	**1.00**

- Joint probabilities appear in the center of the table.
- Marginal probabilities appear in the margins of the table.

Conditional Probability (4 of 4)

Example: Promotion status of police officers over the past two years

Event A = An officer is promoted

Event M = The promoted officer is a man

$P(A|M)$ = An officer is promoted given that the officer is a man

$$P(A|M) = \frac{P(A \cap M)}{P(M)}$$

From the table we know: $P(A \cap M) = 0.24$

$P(M) = 0.8$

$P(A|M) = 0.24/0.8 = 0.3$

Multiplication Law (1 of 3)

- The multiplication law provides a way to compute the probability of the intersection of two events.
- The law is written as:

$$P(A \cap B) = P(B)P(A|B)$$

Multiplication Law (2 of 3)

Example: Newspaper circulation department

Event D = A household subscribes to the daily edition.

Event S = The household already holds a subscription to the Sunday edition.

Given:

$$P(D) = 0.84$$

$$P(S \mid D) = 0.75$$

Multiplication Law (3 of 3)

Example: Newspaper circulation department

What is the probability that the household subscribes to both the Sunday and daily editions of the newspaper?

$$
\begin{aligned}
P(S \cap D) &= P(D)P(S \mid D) \\
&= (0.84)(0.75) \\
&= 0.63
\end{aligned}
$$

Independent Events

- If the probability of event *A* is not changed by the existence of event *B*, we would say that events *A* and *B* are independent.
- Two events *A* and *B* are independent if:

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B)$$

Multiplication Law for Independent Events (1 of 3)

- The multiplication law also can be used as a test to see if two events are independent.
- The law is written as:

$$P(A \cap B) = P(A)P(B)$$

Multiplication Law for Independent Events (2 of 3)

Example : Use of credit card for purchase of gasoline

From past experience, it is known that 80% of customers use credit cards for the purchase of gasoline. The service station manger wants to determine the probability that the next two customers purchasing gasoline will each use a credit card.

Multiplication Law for Independent Events (3 of 3)

Example : Use of a credit card for purchase of gasoline

Event A: First customer uses a credit card

Event B: The second customer uses a credit card

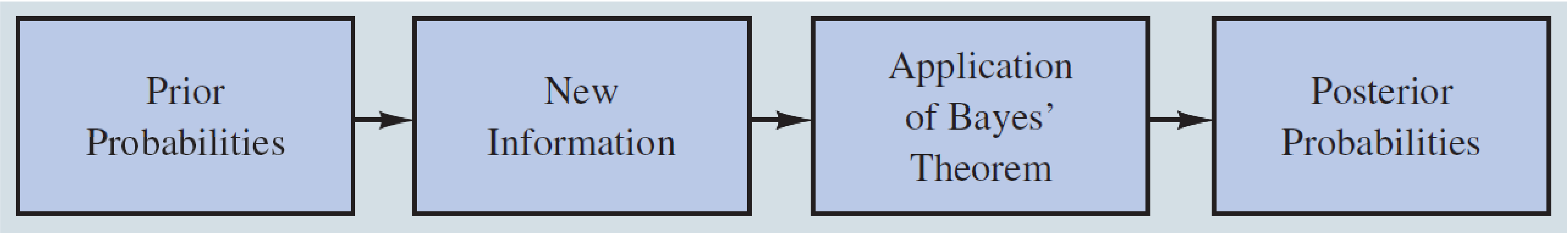
$$
\begin{aligned}
P(A \cap B) &= P(A)P(B) \\
&= (0.8)(0.8) \\
&= 0.64
\end{aligned}
$$

Mutual Exclusiveness and Independence

- Do not confuse the notion of mutually exclusive events with that of independent events.
- Two events with nonzero probabilities cannot be both mutually exclusive and independent.
- If one mutually exclusive event is known to occur, the other cannot occur; thus, the probability of the other event occurring is reduced to zero (and they are therefore dependent).
- Two events that are not mutually exclusive might or might not be independent.

Bayes' Theorem

- Often we begin probability analysis with initial or prior probabilities.
- Then, from a sample, special report, or a product test we obtain some additional information.
- Given this information, we calculate revised or posterior probabilities.
- Bayes' theorem provides the means for revising the prior probabilities.



Prior Probabilities

Example: Two supplier case

A manufacturing firm receives shipments of parts from two different suppliers. The quality of purchased parts varies with the source of supply.

Let:

Event A_1: The part is from supplier 1

Event A_2: The part is from supplier 2

From past experience $P(A_1) = .65, \quad P(A_2) = .35$

New Information

<u>Example:</u> Two supplier case

From historical data, the quality ratings of two suppliers is as below:

Supplier	Percentage of good parts	Percentage of bad parts
Supplier 1	98%	2%
Supplier 2	95%	5%

Conditional Probabilities

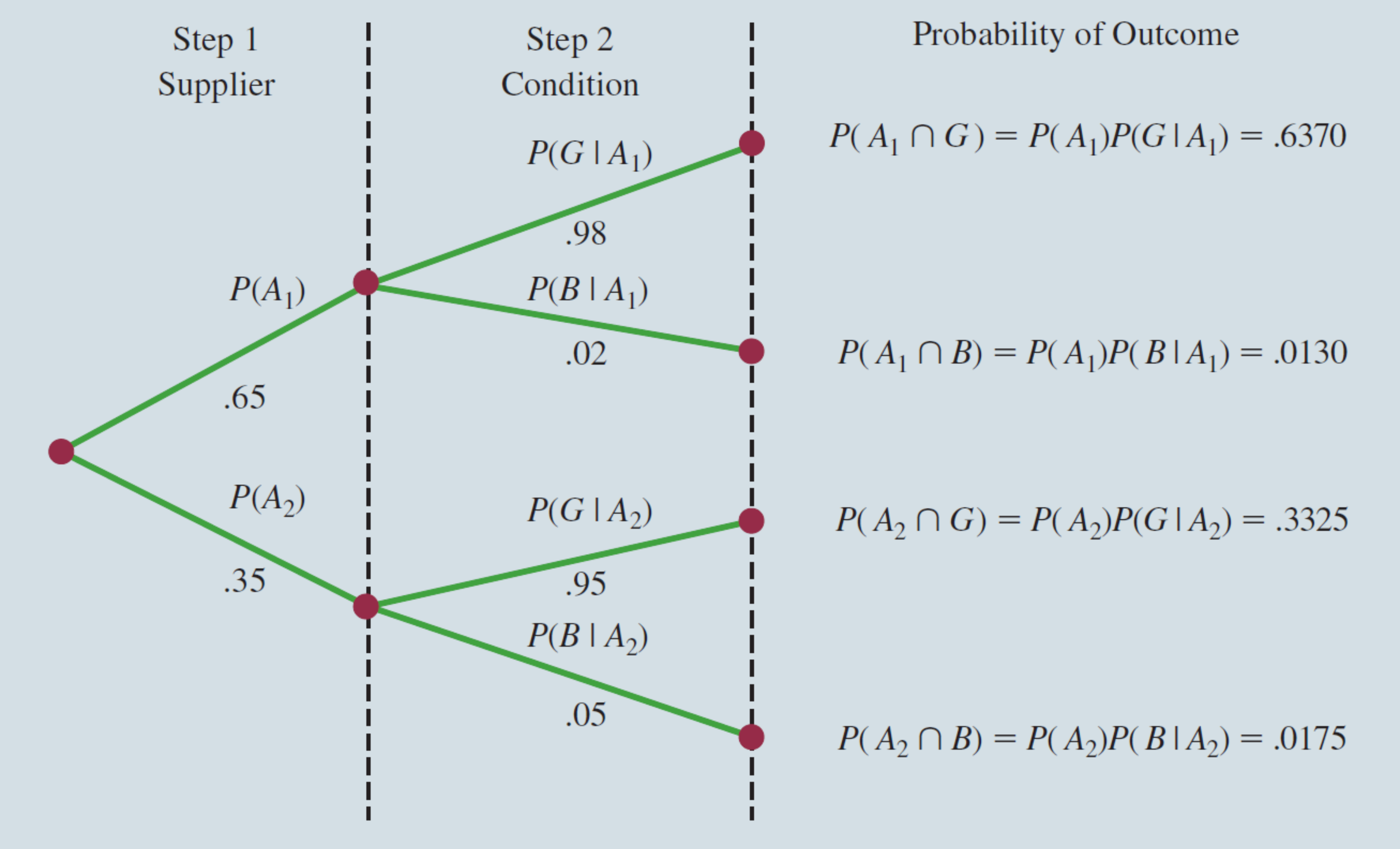
Example

From the prior probabilities and the new information, we have conditional probabilities as below:

$$P(G|A_1) = .98 \qquad P(B|A_1) = .02$$

$$P(G|A_2) = .95 \qquad P(B|A_2) = .05$$

Tree diagram (2 of 2)



Posterior Probability (1 of 3)

- To find the posterior probability that event A_i will occur given that event B has occurred, we apply Bayes' theorem.

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)}$$

- Bayes' theorem is applicable when the events for which we want to compute posterior probabilities are mutually exclusive and their union is the entire sample space.

Posterior Probability (2 of 3)

Example: Two supplier case

Given the part received is bad, we can revise the prior probabilities as below:

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{(.65)(.02)}{(.65)(.02) + (.35)(.05)}$$

$$= .4262$$

Posterior Probability (3 of 3)

Example: Two supplier case

In this application we started with a probability of .65 that a part selected at random was from supplier 1. However, given the information that the part is bad, the probability that the part is from supplier 1 drops to .4262.

In fact, if the part is bad, there is more than 50% chance that it came from supplier 2; that is, $P(A_2 \mid B) = .5738$

Bayes' Theorem: Tabular Approach (1 of 5)

- Step 1

  Prepare the following three columns:

  - Column 1: The mutually exclusive events for which posterior probabilities are desired
  - Column 2: The prior probabilities for the events
  - Column 3: The conditional probabilities of the new information *given* each event

Bayes' Theorem: Tabular Approach (2 of 5)

- Step 2

  Prepare the fourth column:

  - Compute the joint probabilities for each event and the new information *B* by using the multiplication law.
  - Multiply the prior probabilities in column 2 by the corresponding conditional probabilities in column 3. That is, $P(A_i|B) = P(A_i)P(B|A_i)$.

Bayes' Theorem: Tabular Approach (3 of 5)

- Step 3
  - Sum the joint probabilities in column 4.
- Step 4
  - In column 5, compute the posterior probabilities using the basic relationship of conditional probability.

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

The joint probabilities $P(A_i|B)$ are in column 4 and the probability $P(B)$ is the sum of column 4.

Bayes' Theorem: Tabular Approach (4 of 5)

Example: Two Supplier case

Events (A_i)	Prior probabilities $P(A_i)$	Conditional probabilities $P(A_i \mid B)$	Joint probabilities $P(A_i \cap B)$	Posterior probabilities $P(A_i \mid B)$
A_1	.65	.02	.0130	$.0130/.0305 = .4262$
A_2	.35	.05	.0175	$.0175/.0305 = .5738$
	1.00		$P(B) = .0305$	1.0000

Bayes' Theorem: Tabular Approach (5 of 5)

- Inference from the table
  - There is a .0130 probability that the part came from supplier 1 and is bad.
  - There is .0175 probability that the part came from supplier 2 and is bad.
  - The overall probability of finding a bad part from the combined shipments of two suppliers is .0305.